Bernoulli's Equation

The nonlinear differential equation

$$\frac{dy}{dx} + P(x)y = Q(x) \cdot y^n \quad — ①$$

Where n is constant but necessary an integer, known as Bernoulli's equation.

When $n=0$ and $n=1$, the equation is already linear.

The substitution $v = y^{1-n}$ reduces Bernoulli's equation to a linear equation.

Equation ① can be written as

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q(x) \quad — ②$$

$$\therefore \frac{1}{1-n}\frac{dv}{dx} = y^{-n}\frac{dy}{dx}$$

$$\therefore \frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x) \quad — ③$$

Which is linear equation.

Ex① Solve the differential equation

$$x^2y' + 2xy - y^3 = 0 \quad \text{— (1)}$$

Solution:→

$$x^2 y' + 2xy = y^3$$

$$\Rightarrow x^2 \frac{1}{y^3}\frac{dy}{dx} + 2x y^{-2} = 1 \quad \text{— (2)}$$

and setting $v = y^{-2}$, we have

$$\frac{dv}{dx} = -2y^{-2-1}\frac{dy}{dx}$$

$$\Rightarrow -\frac{1}{2}\frac{dv}{dx} = y^{-3}\frac{dy}{dx}$$

∴ (2) becomes,

$$-\frac{x^2}{2}\frac{dv}{dx} + 2xv = 1$$

$$\therefore -\frac{x^2}{2}\frac{dv}{dx} + 2xv = 1. \quad \text{— (3)}$$

Which is linear differential equation

Solving the corresponding homogeneous equation.

$$-\frac{x^2}{2}\frac{dv}{dx} + 2xv = 0$$

$$\Rightarrow \frac{dv}{v} = 4\frac{dx}{x}$$

$$\therefore v = cx^4 \quad \text{— (4)}$$

3.

Let the general solution of (1) is of the form

$$v(x) = A(x)x^4$$

Substituting the value of v in (3), gives

$$\frac{dv}{dx} = 4Ax^3 + A'x^4$$

$$\therefore -\frac{x^2}{2}(4Ax^3 + A'x^4) + 2xAx^4 = 1.$$

$$\Rightarrow \quad A' = -\frac{2}{x^6}$$

integrating

$$A = \frac{2}{5}x^{-5} + c_1$$

Where c_1 is arbitrary constant

Hence the solution of (1) is

$$v = \left(c_1 + \frac{2}{5}x^{-5}\right)x^4$$

$$\therefore y = \left(\frac{2}{5}x^{-1} + c_1x^4\right)^{-\frac{1}{2}}$$

Ans.

Example(2) Solve the differential equation

$$x^2y' + 2xy - y^3 = 0 \quad \text{---} (1)$$

Solution:→

$$\therefore y' + \frac{2}{x}y = \frac{y^3}{x^2} \quad - (2)$$

We have $p(x) = \frac{2}{x}$, $g(x) = \frac{1}{x^2}$

putting $y = u(x)\cdot v(x)$, we get

$$uv' + u\left(v' + \frac{2v}{x}\right) = \frac{(uv)^3}{x^2} \quad \text{---} (3)$$

Let $v(x)$ is a particular solution of

$$v' + \frac{2}{x}v = 0$$

the separating the variables, we find

$$\frac{dv}{v} = -\frac{2}{x}dx, \quad v(x) = x^{-2}$$

on substituting $v(x)$ in (3), we have

$$x^{-2}u' = u^3 x^{-8}$$

$$\text{or, } \frac{du}{u^3} = \frac{dx}{x^6}$$

Integrating both sides, yields

$$u^{-2} = \frac{2}{5}x^{-5} + c$$

$$\Rightarrow u = \left(c + \frac{2}{5}x^{-5}\right)^{-\frac{1}{2}}$$

$\therefore$ general solution is $y = \left(cx^4 + \frac{2}{5}x^{-1}\right)^{-\frac{1}{2}}$ Ans.